Thermodynamics of the slow solutions to the gas-piston equations

D. Chiuchiù[1] G. Gubbiotti[2]

[1]NiPS Laboratory, Università degli Studi di Perugia

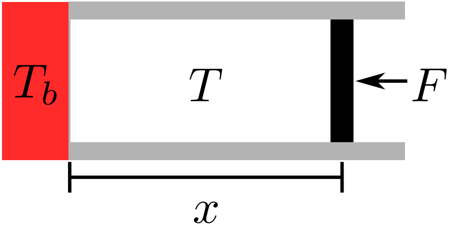
[2]Università di Roma Tre

Gubbio, Micro Energy Conference
6/7/17

Outline

1. The gas-piston-reservoir problem
2. Slow solutions to the gas-piston-reservoir system
3. Thermodynamics of the slow solution of the gas-piston equations

Gas-piston nonequilibrium thermodynamics



x: piston position

T: gas temperature

T_b: reservoir temperature

F: external force

If F and T_b changes, how do T and x evolve?
How much energy do I have to invest?

Standard thermodynamics:

- Describes the final equilibrium state.
- No dynamical description.

Dynamical equations for the gas-piston system

$$\frac{T_b}{F}\dddot{x} + \left[2\frac{\sqrt{T_b}\sqrt{2}\varepsilon}{\sqrt{\pi}} - \frac{T_b\dot{F}}{F^2} + \frac{\dot{T}_b}{F} + \frac{\sqrt{T_b}\sqrt{2}}{\sqrt{\pi}\varepsilon}\right]\ddot{x} + \left[3F + \frac{\sqrt{2}\varepsilon\dot{T}_b}{\sqrt{\pi}\sqrt{T_b}} + 4\frac{F}{\pi}\right]\dot{x}$$

$$+ \left[\dot{F} + \frac{\sqrt{2}F^2}{\sqrt{\pi}\varepsilon\sqrt{T_b}}\right]x - \frac{\sqrt{2}F\sqrt{T_b}}{\sqrt{\pi}\varepsilon} = 0$$

$$T = \frac{T_b}{F}\ddot{x} + 2\sqrt{\frac{2}{\pi}}\sqrt{T_b}\varepsilon\dot{x} + Fx$$

- Derived in *PRE* **91**, 032128.
- System close to dynamical equilibrium.
- $\varepsilon = \sqrt{\frac{Nm}{M}}$ only relevant dynamical parameter.
- $x(t) = \sum_{n=0}^{N} \varepsilon^n x_n(t_0, \ldots, t_N) + \mathcal{O}\left(\varepsilon^{N+1}\right)$

Finite-time isothermal compression

Heat exchange for a finite-time isothermal compression

$$\Delta Q(\tau_{pr}) = \ln(1+f) - \frac{2f^2\left[(1+f)^2 - \frac{1}{2}\right]\left[(1+f)+\frac{1}{2}\right]}{{\tau_{pr}}^2(1+f)^4} + \frac{2f\varepsilon\sqrt{\frac{2}{\pi}}\left[(\pi+\frac{3}{2})(1+f)^2 - \frac{\pi+2}{4}\right]}{\tau_{pr}(1+f)^2} + \mathcal{O}\left(\varepsilon^3\right)$$

Main results:

- Analytic expression rare in the literature.
- Consistent with the second law of thermodynamics.
- Also obtainable with a *slow solution*.

Slow solution and related ansatz

$$x(t) = \sum_{n=0}^{\infty} \varepsilon^n \, x_n(\varepsilon t)$$

- Only the slowest timescale εt appears.
- Transient behaviors automatically ruled out.
- Stationary solution for periodic F and constant T_b.

Slow solution ansatz

Slow solution to study stationary regimes of the gas-piston for changes in F and T_b *slow but not quasi-static*.

Recursive relation for the slow solution

The $x_n(\varepsilon t)$ satisfy the recursive relation

$$x_n(\varepsilon t) = L_2 x_{n-2}(\varepsilon t) + L_4 x_{n-4}(\varepsilon t)$$

$$x_0(\varepsilon t) = \frac{T_b}{F} \quad x_1(\varepsilon t) = 0 \quad x_3(\varepsilon t) = 0$$

$$x_2(s) = \frac{\sqrt{T_b}}{F^2}\left[\alpha_1 \frac{F' T_b}{F} - \alpha_2 T_b'\right] + \frac{T_b}{F^3}\left[\frac{T_b F''}{F} - T_b''\right] + 2\frac{T_b F'}{F^4}\left[T_b' - \frac{T_b F'}{F}\right]$$

$$L_2 = -\frac{T_b}{F^2}\frac{\mathrm{d}^2}{\mathrm{d}(\varepsilon t)^2} - \sqrt{\frac{1}{2\pi}}\frac{\sqrt{T_b}\,(3\pi+4)}{F}\frac{\mathrm{d}}{\mathrm{d}(\varepsilon t)} - \sqrt{\frac{\pi}{2}}\frac{F'\sqrt{T_b}}{F^2}$$

$$L_4 = -\sqrt{\frac{\pi}{2}}\frac{T_b^{3/2}}{F^3}\frac{\mathrm{d}^3}{\mathrm{d}(\varepsilon t)^3} - \frac{T_b'}{F^2}\frac{\mathrm{d}}{\mathrm{d}(\varepsilon t)} - \left[\frac{2T_b}{F^2} + \sqrt{\frac{\pi}{2}}\frac{\sqrt{T_b}}{F^2}\left(\frac{T_b}{F}\right)'\right]$$

Importance of the recursion relation

- Simple operations. Compute $x(t)$ to any desired precision.
- $x_{2n+1}(\varepsilon t) = 0 \; \forall n \rightarrow$ High precision with few iterations.
- Valid for many slow F and T_b. Attractive solution to the problem.

Slow solution good for thermodynamic cycles in stationary regime

Thermodynamics of the slow solution

Phys. Rev. E., 94, 042106

Heat exchange for a slow thermodynamic cycle in stationary regime

$$\Delta Q(\varepsilon t) = \sum_{n=0}^{\infty} \varepsilon^{2n} \Delta Q_{2n}(\varepsilon t)$$

$$\Delta Q_0(\varepsilon t) = -\left(F\, x_0\right)\Big|_0^{\varepsilon t} - \int_0^{\varepsilon t} F\, x_0'\, \mathrm{d}\sigma$$

$$\Delta Q_{2n}(\varepsilon t) = -\int_0^{\varepsilon t} F\, x_{2n}'\, \mathrm{d}\sigma - \frac{1}{2} \sum_{l=0}^{n-1} \left(x_{2l}'\, x_{2n-2l-2}'\right)\Big|_0^{\varepsilon t} - \left(F\, x_{2n}\right)\Big|_0^{\varepsilon t}$$
$$- \sqrt{\frac{8}{\pi}} \left(\sqrt{T_b}\, x_{2n-2}'\right)\Big|_0^{\varepsilon t} - \left(\frac{T_b}{F}\, x_{2n-2}''\right)\Big|_0^{\varepsilon t}.$$

Importance of the series for the heat

- Formal expression of ΔQ for nonequilibrium processes.
- Series well defined at every ε order.
- Evaluate ΔQ a priory with the sole knowledge of F and T_b.
- Characterize situations with high efficiency and low power output.
- Study full in detail the Ericsson-like thermodynamic cycles.

Conclusions

- Recursive relation for the gas-piston dynamical observables.
- Analytic structure of the heat exchanged with the reservoir.
- Analysis of thermodynamic cycles in nonequilibrium situations.